Theorem 1 (Existence of Monotone Subseq). Let (a_n) be any sequence (of real numbers). Then it has a monotone subsequence.

Pf. We call a natural ~~number~~ $m \in \mathbb{N}$ a peak-index (a_m is then called a peak-term) for the seq (a_n) if

$$(*) \qquad a_m \geq a_{m+j} \quad \forall j \in \mathbb{N}.$$

("走下坡" or "無更好的日子" if a_m is one's income at mth year)

Thus either there are (at most) finitely many peak-indices or there are infinitely many peak-indices.

Case 1: There are only finitely many peak-indices so $\exists N \in \mathbb{N}$ s.t. $\nexists$ peak-indices greater than N: if $m > N$ then m is not a peak-index, i.e.

$$(**) \qquad a_m < a_{m+j} \text{ for some } j \in \mathbb{N}.$$

"希望在人間"

In particular, we pick $n_1 = N+1$ so $\exists n_2 > n_1$ such that

$$a_{n_1} < a_{n_2}$$

(applying m to n_1 and $n_2 = n_1 + j$ for suitable j).

Repeating the same to n_2 for m we get some $n_3 > n_2$ s.t. $a_{n_2} < a_{n_3}$. Inductively we have a strictly increasing seq $(n_k)_{k \in \mathbb{N}}$ of natural numbers s.t.

$$a_{n_k} < a_{n_{k+1}} \quad \forall\ k \in \mathbb{N}$$

so (a_{n_k}) is a (strictly) increasing subseq of (a_n).

Case 2. There are infinitely many peak-indices, so we have

$$n_1 < n_2 < n_3 < \cdots$$

with each of them being a peak-index. By definition of peak, one then has

$$a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \cdots$$

so (a_{n_k}) is a decreasing subseq of (a_n).

Th 2 (Bolzano-Weierstrass). Let (x_n) be a bounded seq (so $\exists$ real $a < b$ s.t. $a \leq x_n \leq b\ \forall n \in \mathbb{N}$). Then (x_n) has a convergent subseq.

Pf. By Th1, $\exists$ a monotone subseq (x_{n_k}) of (x_n). Since (x_{n_k}) is also bounded (with lower bound a & upper bound b), it must be convergent by MCT.

Note. $\lim_k x_{n_k} \in [a,b]$ (by the order-preserving property for limits). Thus Th2 can alternatively be stated as follows:

Th2' (Bolzano-Weierstrass Th). Let (x_n) be a sequence in a bounded closed interval $[a,b]$. Then $\exists$ a convergent subseq (x_{n_k}) such that

$$\lim_k x_{n_k} \in [a,b].$$

Note. The B-W theorem can be alternatively proved by Nested Interval Th. (& "Bisection Technique").

Th3 (Cauchy Criterion). Sequence (x_n) converges iff it is Cauchy.

Proof. We show the sufficient part (as the Nece. Part already noted).

Let (x_n) be Cauchy. In particular, $\exists$ $N_0 \in \mathbb{N}$ s.t.

$$|x_m - x_n| < 1 \quad \forall\, m, n \geq N_0.$$

Let $M := \max\{|x_{N_0}| + 1, |x_1|, \dots, |x_{N_0}|\}$

Then, as before, $|x_n| \leq M \;\forall n$, that is (x_n) is bounded, and hence it has a convergence subseq (x_{n_K}):

$$x := \lim_{K} x_{n_K}.$$

We show, with the aid of Cauchy property, that

$$x = \lim_{n} x_n$$

To do this, let $\varepsilon > 0$. Take $N \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon/2 \quad \forall\, m, n \geq N;$$

and also $\exists\, K \in \mathbb{N}$ s.t

$$|x_{n_k} - x| < \varepsilon/2 \quad \forall\, k \geq K.$$

Take a natural number $k \geq N, K$ (so also $n_k \geq k \geq N$). Note then that, for this k,

$$|x_n - x_{n_k}| < \varepsilon/2 \quad \forall\, n \geq N$$

$$|x_{n_k} - x| < \varepsilon/2$$

and it follows from the Δ-inequality that

$$|x_n - x| < \varepsilon \quad \forall\, n \geq N.$$

i.e. $\lim x_n = x$.